U.G. 6th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-13

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Find the solution of the equation $|z| - z = 1 + 2i$.

b) Find the value of $(1+i)^{10} + (1-i)^{10}$.

c) If $f(z) = z^2 + 2$, then find the minimum value of $|f(z)|$ over the closed region $|z| \le 1$.

d) At what point the function $f(z) = |z|^2 + i\overline{z} + 1$ is differentiable?

e) If an analytic function $f(z)$ is such that $\operatorname{Re}\{f'(z)\} = 2y$, $f(1+i) = 2$, then find the imaginary part of $f(z)$.

f) Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the path $y = x$.

g) Find the value of $\int_C \frac{\cos z}{z(z^2+9)}\,dz$, where $C : |z| = 2$.

h) Show that open interval (0,1) of reals with usual metric is an incomplete metric space.

i) Either prove or disprove: In a metric space (X, d), if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ then $\{x_n\}$ is a Cauchy sequence in *X*.

j) Does a homeomorphism preserve completeness? Justify.

k) Either prove or disprove: Every connected proper subset of *R* with usual metric is contained in some compact subset of *R*.

l) Give an example of a subset in a metric space which is bounded without being compact.

m) Show that the set $X = R$ with the metric

$$d(x, y) = \frac{|x-y|}{1+|x-y|}$$ is bounded.

n) Give example of subsets of R which are disjoint but not separated.

o) Show that $f(z) = |z^2|$ is continuous everywhere but nowhere differentiable except at the origin.

2. Answer any **four** questions: 5×4=20

a) Let $f(z) = u + iv$ be analytic in a domain D and $|f(z)|$ is constant in D. Show that $f(z)$ is constant in D.

b) If $f(z)$ is analytic, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

c) Evaluate $\int_C \frac{z}{(z+i)(9-z^2)} dz$ where C is the circle $|z| = 2$.

d) Prove that a sequentially compact metric space is totally bounded.

e) Let $f : (X, d) \to (Y, \rho)$ be a one – one and onto continuous function, where (X, d) is compact. Show that $f^{-1} : Y \to X$ is continuous.

f) Show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a convergent sequence in real number space with usual metric, and hence obtain $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$.

3. Answer any **two** questions: 10×2=20

a) i) Let $f(z) = \sqrt{|xy|}$. Show that $f'(0)$ does not exist but the C-R equations are satisfied at the origin.

ii) Let f be analytic in a simply connected region R and let α, β be any two points in R. Prove that $\int_\alpha^\beta f(z)dz$ is independent of the path joining α and β in R.

5+5

b) i) Examine if $f(x) = x^2$ is a uniformly continuous function over the space R of all reals with usual metric.

ii) Obtain the closure of the set $\left\{(x, y) : y = sin\frac{1}{x} \text{ } and \text{ } 0 < x \leq 1\right\}$ in R^2 with usual metric. 5+5

c) i) Evaluate $\int_C \frac{1}{(z-1)^3} dz$, where C is the line segment from $z = 1 + i$ to $z = 3 + 2i$.

ii) Let $f : (X, d) \to (Y, \rho)$ be a continuous function where (X, d) is compact. Show that $f(\overline{A}) = \overline{f(A)}$. 5+5

____