2021
MATHEMATICS
[HONOURS]
Paper : VII

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

GROUP–A

1. Answer any **five** questions: 1×5=5

a) Write frequency definition of probability.

b) For any random variable X, if F(x) is the distribution of X and a<b be any two fixed points, then show that

$$P(a < X \le b) = F(b) - F(a).$$

c) State Law of Large numbers.

d) Define consistent estimator of a parameter.

e) Define a covariant tensor of order 2.

f) If the relation $b_j^i v_i = 0$ holds for any arbitrary covariant vector v_i show that $b_j^i = 0$.

g) Define bilinear transformation.

h) For two vectors $\vec{a}$ and $\vec{b}$ define $\vec{a} \times \vec{b}$.

[Turn over]

GROUP–B

2. Answer any **ten** questions: 2×10=20

a) If $|z - 2 + i| \le 2$, then show that

$$\sqrt{5} - 2 \le |z| \le \sqrt{5} + 2.$$

b) Prove that S_j^i is a mixed tensor of rank 2.

c) If $P(A) > P(B)$ then show that

$$P(A | B) > P(B | A)$$

d) If X_1 and X_2 are two indenpendent random variables that follow Poisson Distribution with parameters λ_1 and λ_2, prove that $(X_1 + X_2)$ also follow Poisson Distribution with parameter $\lambda_1 + \lambda_2$.

e) State De Moivre-Laplace limit theorem.

f) Show that the sample mean based on a simple random sample with replacement, is an unbiased estimator of the population mean.

g) If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then show that $\vec{A} = (\vec{A}.\hat{i})\hat{i} + (\vec{A}.\hat{j})\hat{j} + (\vec{A}.\hat{k})\hat{k}$.

h) Find the unit tangent vector to any point on the curve $x = t^2 - t, y = 4t - 3, z = 2t^2 - 8t$. Hence find the unit tangent at the point where t = 2.

i) Show that $\nabla^2\left(\frac{1}{r}\right)=0$, where

$r=\sqrt{x^2+y^2+z^2}$ and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

j) For any vector $\vec{A}$, show that

$\int \vec{A}\times\frac{d^2\vec{A}}{dt^2}\,dt=\vec{A}\times\frac{d\vec{A}}{dt}+\vec{C}$

where $\vec{C}$ is a vector constant.

k) If z_1, z_2 and z_3 are three complex numbers, then prove that

$z_1\,\mathrm{Im}(\bar{z}_2 z_3)+z_2\,\mathrm{Im}(\bar{z}_3 z_1)+z_3\,\mathrm{Im}(\bar{z}_1 z_2)=0$.

l) Show that the transformation $\omega=\frac{2z+3}{z-4}$ transforms the circle $x^2+y^2-4x=0$ into the straight line $4u+3=0$ where $z=x+iy$ and $w=u+iv$.

GROUP–C

3. Answer any **five** questions: 6×5=30

a) If (X,Y) is uniformly distributed over the semicircle bounded by $y=\sqrt{1-x^2}$ and $y=0$, find $E[X/Y]$ and $E[Y/X]$. Also verify $E\{E(X/Y)\}=E\{X\}$ and $E\{E(Y/X)\}=E\{Y\}$.

b) If the probability distribution of a discrete random variable X is given by

$P(X=x)=Ke^{-t}\left(1-e^{-t}\right)^{x-1}\ x=1,2,.....\infty$, find the value of K and also the mean and variance of X.

c) Show that the function $u=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$ satisfies Laplace's equation and determine the correponding analytic function $f(z)=u+iv$.

d) Determine the bilinear transformation which transforms the circle $|z|<\rho$ on to the circle $|\omega|<\rho'$.

e) If $a_{pqr}x^p x^q x^r=0$ for all values of the independent variables $x^i, i=1,2.....N$ and a_{pqr}'s are constants, show that

$$a_{ijk}+a_{ikj}+a_{jik}+a_{jki}+a_{kij}+a_{kji}=0$$

f) Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$, where $A=18z\hat{i}-12\hat{j}+3y\hat{k}$ and S is the part of the plane $2x+3y+6z=12$ which is located in the first octant.

g) Prove that $\vec{b}\cdot\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$

where $r=\sqrt{x^2+y^2+z^2}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

h) The probability density function of the velocity

V of a gas molecule is given by

$$f_v(v)=\begin{cases} kv^2e^{-av^2} & v>0 \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant depending on its mass and k in an appropriate constant. Find the constant k and show that the kinetic energy $Y=\frac{1}{2}mV^2$ is a random variable having Gamma distribution.

GROUP–D

4. Answer any **three** questions: 15×3=45

a) i) If the joint probability density function of (X, Y) is given by

$$f(x,y)=\frac{9(1+x+y)}{2(1+x)^4(1+y)^4},\ x>0,\ y>0$$

find the marginal densities of X and Y. Are they independent?

ii) If a, b, c are positive constants, show that the correlation co-efficient between aX+bY and cY is

$$\frac{a\rho\sigma_x+b\sigma_y}{\sqrt{a^2\sigma_x^2+b^2\sigma_y^2+2ab\rho\sigma_x\sigma_y}}.$$

iii) If the joint distribution of X and Y is the general bivariate normal distribution and

$$U=\frac{X-mx}{\sigma_x},\ V=\frac{1}{\sqrt{1-\rho^2}}\left\{\frac{Y-my}{\sigma_Y}-\rho\frac{X-mx}{\sigma_X}\right\}$$

then show that U and V are independent standard normal variate.

b) i) If X is a binomial (n, p) variate then show that the moment generating function of X is $(q+pe^t)^n$, where $q=1-p$.

Hence show that if $Z=\frac{X-np}{\sqrt{npq}}$, then for $n\to\infty$, Z will follow standard normal distribution.

ii) When $n\to\infty$ and np=constant, then show that Binomial (n, p) distribution tends to Poisson distribution.

iii) A Random variable is exponentially distributed with parameter 1. Use Tchebycheff's inequality to show that $P(-1\le X\le 3)\ge\frac{3}{4}$. Find the actual probability also. [Density of exponential distribution is $f(x)=\lambda e^{-\lambda x},\lambda>0$].

c) i) A computer while calculating correlation coefficient between two variables X and Y from 25 pairs of observations obtained the following results:

$N=25,\ \Sigma X=125, \Sigma X^2=650,$

$\Sigma Y = 100$, $\Sigma Y^2 = 460$, $\Sigma XY = 508$.

It was however, discovered at the time of checking that two pairs of observations were not correctly copied. They were taken as (6, 14) and (8, 6) while the correct values were (8, 12) and (6, 8). Prove that the correct value of the correlation coefficient should be 2/3.

ii) A tossed a biased coin 50 times and get head 20 times while B tossed it 90 times and got head 40 times. Find the maximum likelihood estimate of the probability of getting head when the coin is tossed, after constructing the likelihood function.

iii) The mean yield of wheat from a district A was 210 lbs with S.D. =10 lbs per acre from a sample of 100 plots. In another district B, the mean yield was 220 lbs with S.D.=12 lbs from a sample of 150 plots. Assuming that the standard deviation of yield in the entire state was 11lbs, test whether there is any significant difference between the mean yield of crops in the two districts. Test it at 1% level.

[Given $P(0<Z<2.58)=0.4951$].

d) i) Verify the divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

ii) Verify Stoke's theorem for $\vec{F} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ where S is the surface of the cube $x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 2,\ z = 2$ above the xy-plane.

iii) Show that inner product of the vectors A^p and B_q is an invariant.

e) i) Show that the relation $\omega = \dfrac{iz + 2}{4z + i}$, transforms the real axis in the z-plane to a circle in the ω-plane. Find the centre and the radius of the circle.

ii) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find f(z) subject to the condition $f(\pi/2) = \dfrac{3 - i}{3}$.

iii) Find all circles which are orthogonal to $|z|=1$ and $|z-1|=4$.
