2021
STATISTICS
[HONOURS]
Paper : V

Full Marks : 75 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer all the questions.

1. Answer any **five** questions: 1×5=5

   i) Define *t* distribution with 2 degrees of freedom.

   ii) Define asymptotic efficiency of an estimator *T* based on a sample of *n* observations.

   iii) If the distribution of Y is F with (n_1,n_2) d.f., what is the distribution of $\frac{1}{Y}$?

   iv) Write down an unbiased estimator of σ^2 when $X_1,X_2,\ldots X_n$ are iid observations from a $N(\mu,\sigma^2)$ population where both parameters are unknown.

[Turn over]

   v) What is p-value of a test?

   vi) What is MVUE in the context of estimation theory?

   vii) If U and V are two statistics with $E(U)=3\theta$ and $E(V)=2\theta$, obtain an unbiased estimator of θ.

   viii) What is composite hypothesis? Give an example.

   ix) Describe randomized test.

2. Answer any **six** questions: 2×6=12

   i) Let X be an observation from $f_\theta(x)=\theta e^{-\theta x}, 0<x<\infty, \theta>0$. If (X, 2X) is a confidence interval for $1/\theta$, find its confidence coefficient.

   ii) Suppose $X_1 \sim Bin(n_1,p)$, $X_2 \sim Bin(n_2,p)$ and X_1, X_2 are independent. Find the conditional distribution of X_1 given X_1+X_2.

   iii) If (X_1, X_2) be a random sample of size 2 from a $P(\lambda)$ population, check from the definition whether $T=X_1+X_2$ is sufficient for λ.

   iv) Define an unbiased estimator. Is it unique? Justify.

v) Define an UMP test. Does it always exist? Give example to justify your answer.

vi) Write Polar transformation for n real valued random variables.

vii) Let $X_1, X_2 \sim$ iid Bernoulli(θ). Check whether the statistic $T = X_1 + 2X_2$ is sufficient for θ.

viii) State Rao-Cramer inequality.

ix) For two size α test procedures, which one will you prefer? Define that criterion which you will use to prefer one.

x) Show that if X and Y follow independent Chi-square distributions with m and n degrees of freedom respectively then the distribution of X + Y is Chi-square with m + n degrees of freedom.

3. Answer any **three** questions: 6×3=18

i) Let $X_i \sim N(\mu, \sigma^2)$ independently for $i = 1,2,...,$ n. Find the sampling distribution of

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ where } \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

ii) The random variables X_i(i=1, 2, ...n) are independently distributed, respectively, as $N(0, \sigma_i^2)$.

Let $\tilde{X} = \sum_{i=1}^{n} \frac{X_i}{\sigma_i^2} \Big/ \sum_{i=1}^{n} \frac{1}{\sigma_i^2}$

Show that $\sum_{i=1}^{n} \frac{1}{\sigma_i^2}(X_i - \tilde{X})^2$ is distributed as Chi-square with (n–1) d.f.

iii) a) Show that the Neyman-Pearson test is a function of a sufficient statistic.

b) If $X_1,...X_n$ be iid Cauchy(θ), show that the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is not consistent for θ. Suggest a consistent estimator in this case.

iv) a) Define a consistent estimator and state a set of sufficient conditions for consistency.

b) For two inconsistent estimators of θ, which evaluation criterion is used to prefer any one of them? Define that criterion. 3+3

iv) Let X_1, X_2 and X_3 be a random sample of size three from a uniform $(\theta, 2\theta)$ distribution., where $\theta > 0$. Find MME and MLE of θ. Which will you prefer and why?

4. Answer any **four** questions: 10×4=40

i) a) Define MSE in connection to the theory of estimation.

b) Let the random variable $Y_1, Y_2, \ldots, Y_n$ satisfy $Y_i = \beta x_i + e_i,\ i = 1,2,\ldots n$, where $x_1, x_2, \ldots, x_n$ are fixed constants, and $e_1, e_2, \ldots e_n$ are iid $N(0,\sigma^2), \sigma^2$ is unknown. Find the distribution of Y_i and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Using the result also construct an unbiased estimator of β.

c) On the basis of a random sample of size n drawn from the distribution with p.d.f.

$$f_\theta(x) = \frac{1}{2\theta} e^{-|x|/\theta}, -\infty < x < \infty, \theta > 0,$$

Derive a method of moments estimator for θ. 4+2+4

ii) Obtain the distribution of the sample range based on a sample of size n drawn from a $U(-\theta, \theta)$ distribution.

iii) a) Let X be an observation from the distribution with p.d.f.

$$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|};$$

$$x = -1, 0, 1 \ ; \ 0 \le \theta \le 1$$

Find the MLE of θ. Also show that the estimator $T(X) = \begin{cases} 2 & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$

is an unbiased estimator of θ.

b) Define uniformly minimum variance unbiased estimator of a function $\gamma(\theta)$. Show that such an estimator, if exists, is unique. 5+5

iv) a) Suppose $(X_1, X_2, \ldots X_m)$ and $(Y_1, Y_2, \ldots Y_n)$ are two iid random samples from $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ distributions respectively, where σ_x and σ_y are known. Find a $100(1-\alpha)\%$ confidence interval for the difference between μ_x and μ_y.

b) Suppose X is an observable random variable with its pdf given by $f(x), x \in R$. Consider two functions defined as follows:

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right], \quad f_1(x) = \frac{1}{\pi}\frac{1}{1+x^2},$$

Derive MP level α test for $H_0 : f(x) = f_0(x)$ against $H_1 : f(x) = f_1(x)$. Also compute power of the test. 4+6

v) a) Let $X_1, X_2, \ldots, X_n$ be iid Exponential (γ). Find an unbiased estimator of γ based only on $Y = \min(X_1, X_2, \ldots, X_n)$

b) State Rao-Blackwell theorem. On the basis of a sample of size n from Bernoulli(θ), find the MVUE of θ^2. Show every step required. 5+(1+4)

vi) a) Let $X_1, X_2, \ldots X_n$ be a random sample from a univariate normal distribution with mean θ and variance σ^2, where both are unknown. Find a likelihood ratio test for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$.

b) Define Likelihood Ratio Test (LRT) statistic and show that it lies in (0,1).

6+4

———————