U.G. 5th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-01B

(Point Set Topology)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

1. Answer any **ten** questions: $2\times10=20$

i) Show that set of all irrational numbers is uncountable.

ii) Let (X, τ) be a topological space and $\mathcal{B}$ be a subcollection of τ such that every member of τ is a union of some members of $\mathcal{B}$. Show that $\mathcal{B}$ is an open basis for the topology τ on X.

iii) Let (X,d) be a metric space and A be a fixed but arbitrary point of X. Prove that the function $f:(X,\tau(d))\to\mathbf{R}$ defined by $f(x)=d(x, A)$ is a continuous function.

iv) Give an example to show that the intersection of two compact subsets may not be compact.

v) Show that a continuous mapping from a countable connected space to the real line is constant.

vi) Give an example of a connected space which is not locally connected.

vii) Show that the characteristic function of a subset A of a topological space X is continuous on X if A is clopen in X.

viii) State Alexander's subbase theorem.

ix) State Baire Category theorem.

x) State Schroeder-Bernstein theorem.

xi) Give an example of a space which is locally compact but not compact.

xii) Justify the statement : A subset of a metric space is compact if and only if it is closed and bounded.

xiii) Prove that $Gl(n,\mathbb{R})$ is disconnected when it is equipped with subspace topology of $\mathbb{R}^{n^2}$.

xiv) Prove that $Gl(2,\mathbb{C})$ is non-compact when it is equipped with subspace topology of $\mathbb{C}^4$.

xv) Show that a function $f : X \to Y$ is continuous, iff $f : (X) \to f(X)$ in continuous where X, Y are two topological spaces and $f(X)$ is a subspace of y.

2. Answer any **four** questions: 5×4=20

i) Show that union of any family of connected sets, no two of which are separated, is a connected set.

ii) a) Show that real line **R** is homeomorphic to the subspace $\mathbf{R} \times \{0\}$ of the Euclidean plane.

b) Show that every continuous function $f : [0,1] \to [0,1]$ has at least one fixed point.

iii) Show that a subset of **R**, consisting of at least two points is connected if and only if it is an interval.

iv) Show that a function $f : (X, \tau) \to (Y, \tau')$ is open if and only if for each point $x \in X$, and each open nbd U of x, there exists a nbd W of $f(x)$ in Y such that $W \subseteq f(U)$.

v) Let (X, τ) be the product of a family of topological spaces $\{(X_i, \tau_i) : i = 1, 2, ..., n\}$ and $p_i : X \to X_i$ denote the i-th projection map. Then show that

a) p_i is an open map for each i.

b) The product topology τ is the smallest topology on X such that each projection map p_i is continuous.

vi) Show that any compact metric space is complete.

3. Answer any **two** questions: 10×2=20

i) a) Show that in a topological space X, a subset A of X is dense if and only if every open set in X intersects A.

b) Let (Y, τ_Y) be a subspace of a topological space (X, τ). Then show that F is closed in (Y, τ_Y) if and only if $F = Y \cap K$ for some set K closed in (X, τ). 5+5=10

ii) a) Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function. If $\{x_n\}$ is a sequence converging to x show that $\{f(x_n)\}$ converges to $f(x)$. Is the converse true? (Give an example in case it is false).

b) Show that a map $f : (X, \tau) \to (Y, \tau')$ is continuous if and only if $f^{-1}(K)$ is closed in X, for every closed set K in Y.

5+5=10

iii) a) Show that open subset of a locally connected space is locally connected.

b) Justify the statement : if every real valued continuous function from a topological space X is bounded then X is necessarily compact.

c) Show that a space (X, τ) is compact if and only if every basic open cover of X has a finite subcover. 3+3+4=10

iv) a) Show that a real valued continuous function on a compact space (X, τ) attains its least and greatest values.

b) Show that in a topological space (X, τ) (A) each component of X is closed. (B) Two different components of X are disjoint. (C) Each point in X is contained in exactly one component of X. 4+6=10

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