U.G. 4th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-9

(Multivariate Calculus)

Full Marks : 30 Time : $1\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **five** questions: 2×5=10

a) Test the differentiability of *f(x, y)* at (0, 0), where $f(x, y) = |x| + y$.

b) Show that for the function $f(x, y) = |x| + |y|$, the partial derivatives f_x and f_y do not exist at (0, 0).

c) If $z = f(u-v, v-u)$, show that $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = 0$.

d) Find *a, b, c* so that the vector field $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

e) Give an example of a continuous function *f(x, y)* which does not have partial derivatives of the first order.

f) Evaluate $\int_0^1 \int_0^\pi \int_0^\pi y \sin z \, dx\, dy\, dz$.

g) Find the angle between the gradients of the functions $u = |\vec{r}|$ and $v = \log|\vec{r}|$ at *P*(0, 0, 1).

h) Evaluate $\oint_\Gamma (e^x dx + 2y dy - dz)$ by using Stokes's theorem, where Γ represents the curve $x^2 + y^2 = 4,\ z = 2$.

2. Answer any **two** questions: 5×2=10

a) Verify divergence theorem for the vector function $2xz\hat{i} + y^2\hat{j} + yz\hat{k}$ taken over the surface of the cube bounded by *x* = 0, *x* = 1, *y* = 0, *y* = 1 and *z* = 0, *z* = 1.

b) Show that for the function

$$f(x, y) = \begin{cases} (x^2+y^2)\left(\tan^{-1}\frac{y}{x}\right) & \text{when } x \neq 0 \\ \frac{\pi}{2}y^2 & \text{when } x = 0, \end{cases}$$

$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

c) Use Lagrange's method to find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ whose distances from the straight line $3x + y = 9$ are least and greatest.

d) Show that $\iiint \frac{dx\, dy\, dz}{x^2+y^2+(z-2)^2} = \pi\left(2 - \frac{3}{2}\log 3\right)$, extended over the sphere $x^2 + y^2 + z^2 \leq 1$.

e) A particle moves along the curve $x = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5,$ where t represents time. Find the component of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

3 . Answer any **one** question: 10×1=10

a) i) If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{y^2 - x^2}{2uv(u-v)}$. 5

ii) Verify Green's theorem in a plane for $\oint_C \{(x^2 + xy)dx + xdy\}$, where C is the curve enclosing the region bounded by $y = x^2$ and $y = x$. 5

b) i) Prove that the necessary condition for the extremum of the function *f(x, y, z)*, where *x, y, z* satisfies *g(x, y, z)* = 0 and $\frac{\partial g}{\partial z} \neq 0$, are $\frac{\partial(f,g)}{\partial(x,z)} = 0$ and $\frac{\partial(f,g)}{\partial(y,z)} = 0$. 5

ii) If $f(x,y,z) = a^3x^2 + b^3y^2 + c^3z^2$, where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, then find the maximum or minimum value of *f*, where *a, b, c* are constants. 5

c) i) Evaluate the double integral $\iint_D e^{x^2} dx\, dy$, where the region D is given by $D = \{(x,y) \in \mathbf{R}^2 : 2y \leq x \leq 2$ and $0 \leq y \leq 1\}$. 5

ii) Show that the volume of the solid bounded by the cylinder $x^2 + y^2 = 2ax$ and the paraboloid $y^2 + z^2 = 4ax$ is $\frac{2a^3}{3}(3\pi + 8)$. 5

———————