U.G. 4th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-10

Full Marks : 30 Time : $1\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **five** questions: 2×5=10

   a) Justify the statement : the ring $\mathbb{Z}\times\mathbb{Z}$ is not a field.

   b) Show that a ring is commutative if it has the property that $ab = ca$ implies $b = c$ when $a \neq 0$.

   c) Is $I \cup J$ an ideal in a ring R, if I and J are any two ideals in R? If not, give reasons.

   d) Let R be a ring with unity 1. If the product of any pair of non-zero elements of R is non-zero, prove that $ab = 1$ implies $ba = 1$.

   e) Give an example of a finite non-commutative ring.

   f) Is the ring of integers modulo 8 an integral domain? Justify.

   g) Examine whether $S = \{a + bi \in \mathbb{Z}[i] : a \geq 0\}$ is a subring of $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} : a, b \in \mathbb{Z}, i^2 = -1\}$.

   h) Find the group of units in the ring $\mathbb{Z}_{10}$.

2. Answer any **two** questions: 5×2=10

   a) Let R be a commutative ring with identity $1 \neq 0$. Prove that a proper ideal P of R is prime if and only if R/P is an integral domain.

   b) Prove that every ideal of the ring $\mathbb{Z}$ is of the form $n\mathbb{Z} = (n)$ for some non-negative integer n.

   c) Show that the set of matrices $\left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} | x \in \mathbb{R}^*\right\}$ is a subring of the ring $(M_2(\mathbb{R}, +, \cdot)$. Hence conclude that a subring S of a ring R may have an identity different from the identity of R.

   d) Define a subfield of a field. Show that the field $\mathbb{Q}$ of rational numbers has no proper subfield.

   e) Consider the ring $M_2(\mathbb{Z})$. Let $I = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{Z}\right\}$. Show that I is a left ideal of $M_2(\mathbb{Z})$ but not a right ideal.

3. Answer any **one** question: 10×1=10

a) Let I denote the set of all polynomials in $\mathbb{Z}[x]$ with constant terms zero. Show that I is a prime ideal but not a maximal ideal in $\mathbb{Z}[x]$.

b) Let $n \in \mathbb{Z}$ be a fixed positive integer. Prove that the following conditions are equivalent:

i) n is prime.

ii) $\mathbb{Z}/\langle n \rangle$ is an integral domain.

iii) $\mathbb{Z}/\langle n \rangle$ is a field.

c) If in a ring R, $a^2 = a$ holds for all $a \in R$, prove that the ring is commutative. Further prove that in such a ring, every prime ideal is a maximal ideal .