U.G. 5th Semester Examination - 2021

MATHEMATICS

[PROGRAMME]

Discipline Specific Elective (DSE)

Course Code : MATH-G-DSE-T-1A&B

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations carry their usual meanings.

Answer all the question from selected Option.

Course Code : MATH-G-DSE-T-1A

(Matrices and Linear Algebra)

1. Answer any **ten** questions from the following: 2×10=20

i) Find all non-null matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ whose squares are equal to the null matrix.

ii) Find the dimension of the subspace $\{(x, 2x): x \in R\}$of R^2.

iii) If$= \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$, then find $(A^2 - 3A - 13I)$.

iv) If U and W are two subspaces of a vector space V over F then check whether $U \cup W$ is a subspace or not.

v) Check whether $\{(x,\ x+1): x \in R\}$ is a subspace of R^2 or not.

vi) How many solutions are there for the simultaneous equations: $+y = 1; 3x + 3y = 1$? Justify your answer.

vii) A transformation $T: R^2 \to R^2$ is defined by $T(x, y) = (x, 0)$. Check whether it is linear or not.

viii) If $\alpha = (a, b), \beta = (a + b, b - a)$, and $\gamma = (b, a)$, find suitable scalars p, q such that $p\alpha + q\beta = \gamma$ in R^2.

ix) If the vectors $(0, 1, a), (1, a, 1), (a, 1, 0)$ of the vector space R^3 be linearly dependent, then find the value of a.

x) Find the eigen values of $A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

xi) If λ be an eigen value of an orthogonal matrix A, then show that $\frac{1}{\lambda}$ is also an eigen value.

xii) If W_1, W_2, W_3, be three subspaces of a vector space V over F, then find the smallest subspace contained in each of the above subspaces.

xiii) Compute the inverse of the matrix: $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

xiv) If $W = \{(x, 2y, 3z) : x, y, z \in R\}$, then show that W is a subspace of R^3.

xv) Show that $Ker\ \phi$, where $\phi : V \to W$, is a linear transformation between two vector spaces V, W over a field F, is a subspace of V.

2. Answer any **four** questions: 5×4=20

i) Show that the vectors $\{(1,2,1), (2,1,0), (1,-1,2)\}$ form a basis of the vector space R^3 over R .

ii) Show that every square matrix can be expressed uniquely as a sum of a symmetric and a skew-symmetric matrix.

iii) Verify Cayley Hamilton theorem for the matrix: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

iv) Prove that the points $(x_i, y_i), i = 1,2,3,$ are collinear, if and only if the rank of the matrix $\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$ be less than three.

v) If $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, then evaluate $(A^3 - A^2 - I)$.

vi) Solve the system of equations:

$$2x - 3y + z = 1, x + 2y - 3z = 4, 4x - y - 2z = 8.$$

3. Answer any **two** questions: 10×2=20

i) a) Show that the planes passing through the origin is a proper subspace of R^3.

b) Let $T : R^3 \to R^3$ be a linear transformation given by

$T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$.

Verify $rank(T) + nullity(T) = 3$. 4+6

ii) a) Show that every orthogonal matrix A can be expressed as $(I + S)(I - S)^{-1}$ by a suitable choice of a real skew-symmetric matrix S, provided (–1) is not an eigen value of A.

b) Diagonalize the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$, by finding a nonsingular matrix P such that the diagonal matrix $D = P^{-1}AP$. 4+6

iii) a) When a system of linear equations is called consistent? Give an example of a system of linear equations which is inconsistent.

b) Apply the rank test to examine if the following system of equations is consistent and if so, then find the complete solution of

$$x + 2y - z = 6, 3x - y - 2z = 3, 4x + 3y + z = 9 .$$

2+8

iv) a) Show that the set

$S = \{(1,0,0), (1,1, 0), (1,1,1), (0,1,0)\}$

spans the vector space R^3 but is not a basis.

b) Find the co-ordinate vector of $\alpha = (2,3,1)$ relative to the basis

$\{(1,1,1), (1,1,0), (0,1,0)\}$.

c) Construct an one-dimensional subspace of R^3 containing the vector (1, 2, 3).

5+3+2

Course Code : MATH-G-DSE-T-1B

(Complex Analysis)

1. Answer any **ten** questions from the following:

2×10=20

a) Show that the function $f(z) = x^2 + y^2$ is not analytic at any point.

b) Use the definition of limit to show that $\lim_{z \to z_0} \bar{z} = \bar{z}_0$.

c) If z_1 and z_2 are any two complex numbers, show that $|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$.

d) Show that an analytic function with constant modulus is constant.

e) Show that the function

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{|z|^2}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

is continuous and satisfies the Cauchy-Riemann equation at z = 0.

f) If $|z_1| = |z_2| = 1$ and $\text{amp}\,(z_1) + \text{amp}\,(z_2) = 0$, then show that $z_1 z_2 = 1$.

g) Show that the function $f(z) = z^3$ is analytic in a domain D of the complex plane C.

h) Prove that $\left| \int_c \frac{dz}{z^2 + 10} \right| \leq \frac{2\pi}{3}$, where C is the circle $C : z(t) = 2e^{it}, (-\pi \leq t \leq \pi)$.

i) If $a = \cos\theta + i \sin\theta$, obtain the value of θ in $[0, \pi]$ such that $a^3 = i$..

j) Show that $\lim_{z \to 2i} \frac{z^2+4}{z-2i} = 4i$.

k) Evaluate $\int_C (3z^2 - 2z)\, dz$, where C is the contour defined by $z(t) = t + it^2, t \in [0,1]$.

l) Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable.

m) Find the domain of convergence of this series

$$\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n.$$

n) Prove that $f(z) = \begin{cases} z \operatorname{Re} z, & z \neq 0 \\ 0 & , z = 0 \end{cases}$ is continuous at $z = 0$.

o) Show that the function $f(z) = \begin{cases} \frac{\bar{z}}{z}, & z \neq 0 \\ 0 & , z = 0 \end{cases}$

is discontinuous at $z = 0$.

2. Answer any **four** questions: 5×4=20

a) Let $f(z) = |z|^2$. Show that the derivative of $f(z)$ exists only at the origin.

b) Find the radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n + 1}$$

and also prove that $(2-z)f(z) - 2 \to 0$ as $z \to 2$.

c) State and prove Liouville's theorem.

d) If a complex function f is differentiable and $|f|$ is constant in a rectangular region D, prove that f is constant in D.

e) Let $u(x, y) = e^x \cos y$. Determine a function $v(x, y)$ such that $f = u + iv$ is analytic.

f) State Cauchy's integral formula. Use this formula to find the value of

$$\int_C \frac{z^3 + 3z - 1}{z^2 - 3z + 2}\, dz$$

where C is the circle $|z| = 3$.

3. Answer any **two** questions: 10×2=20

a) i) If $f(z)$ be defined in some neighbourhood of the point $z_0 = x_0 + iy_0$ and $f(x) = u(x, y) + iv(x, y)$, then show that f is continuous at z_0 if and only if both $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

ii) Check whether for any complex number z, $|e^z| \leq e^{|z|}$ holds or not. 6+4

b) i) Consider the function f defined by

$$f(z) = \begin{cases} 0, & \text{when } z = 0 \\ \frac{x^3 - y^3}{x^2 + y^2} + i\frac{(x^3 + y^3)}{x^2 + y^2} & \text{when } z \neq 0 \end{cases}$$

Show that the function f satisfies the Cauchy-Riemann equation at origin, but f is not differentiable at $z = 0$.

ii) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}.$$ 6+4

c) i) For what values of z does the series

$$\sum_{n=0}^{\infty} (-1)(z^n + z^{n+1})$$

converge and find its sum.

ii) Evaluate $\int_{|z|=3} \frac{3z^4 + 2z - 6}{(z-2)^3}\, dz.$ 5+5

d) i) For what values of z do the function w defined by the following equation cease to be analytic?

$z = -e^{-v}(\cos u + i \sin u)$, $w = u + iv$ and

$z = \sin u \cos(iv) + \cos u \sin(iv)$.

ii) Discuss the convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{2n!} z^n.$$ 6+4
