2022
MATHEMATICS
[HONOURS]
Paper : VI

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

1. Answer any **five** questions: 1×5=5

a) Eliminate the arbitrary function ϕ from $z = e^{my}\phi(x-y)$.

b) Prove that the function $f(x,y)=\sqrt{|xy|}$ is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin.

c) Find the interval of absolute convergence for the series

$$\sum_{n=1}^{\infty}\frac{x^n}{n^n}.$$

d) Show that

$$\lim_{n\to\infty}\int_0^a \phi\left(\frac{\sin nx}{\sin x}\right)dx = \lim_{n\to\infty}\int_0^a \phi\left(\frac{\sin nx}{x}\right)dx.$$

e) If $L[F(t)] = f(p)$ then prove that for $\lambda > 0$,

$$L[F(\lambda t)] = \frac{1}{\lambda} f\left(\frac{p}{\lambda}\right).$$

f) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$ does not exist.

g) Find the radius of convergence of the series

$$x + \frac{x^2}{2^2} + \frac{2!}{3^3}x^3 + \frac{3!}{4^4}x^4 + \ldots..$$

h) Show that $z = f(x^2y)$, where f is differentiable satisfying $x\left(\dfrac{\partial z}{\partial x}\right) = 2y\left(\dfrac{\partial z}{\partial y}\right)$.

2. Answer any **ten** questions: 2×10=20

a) If f is bounded and integrable on [a, b] then show that $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$.

b) Find the partial differential equation of all surfaces of revolution, having z-axis as the axis of revolution.

c) Prove that $x = 0$ is an ordinary point of $(x^2-1)y''+xy'-y=0$, but $x=1$ is a regular singular point.

d) If f is continuous on [0, 1] and if $\int_0^1 x^n f(x)dx = 0$ for $n = 0,1,2,...$ then show that f(x)=0 on [0, 1].

e) Show that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$ is invariant for change of rectangular axes.

f) If a function f is continuous on [0, 1] then show that

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0).$$

g) Locate and classify the singular points of the equation $x^3(x^2-1)y''+2x^4y'+4y=0$.

h) Find the stationary points of the function

$$f(x,y,z)=(x+y+z)^3-3(x+y+z)-24xyz+27.$$

i) If f is bounded and integrable on $[-\pi,\pi]$ and if a_n, b_n are its Fourier coefficients then prove that $\sum_{n=1}^{\infty}(a_n^2+b_n^2)$ converges.

j) Compute $\int_{-1}^{1} f\,dx$, where $f(x)=|x|$.

k) Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist for the following function

$$f(x,y)=\begin{cases}1, \text{ if } xy\neq 0\\ 0 \text{ if } xy=0\end{cases}$$

l) Obtain the differential equation of all conics whose axes coincide with the axes of coordinates.

3. Answer any **five** questions: 6×5=30

a) i) Prove that the set C[0, 1] consisting of all real-valued continuous functions defined on [0, 1] with the function d given by

$$d(f,g)=\int_0^1|f(x)-g(x)|dx\ \forall f,g\in c[0,1]$$

is a metric space.

ii) Show by means of an example, the arbitrary union of closed sets in a metric space is not necessarily a closed set.

4+2

b) Find the power series solution of the equation $(x^2+1)y''+xy'-xy=0$ in powers of x (i.e., about x=0)

c) Test for the convergence of the integral $\int_0^1 x^p \left(\log \frac{1}{x}\right)^q dx$.

d) Prove that, by the transformations $u = x - ct,\ v = x + ct$, the partial differential equation $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial t^2}$ reduces to $\frac{\partial^2 z}{\partial u \partial v} = 0$.

e) i) Prove that the set C[a, b] of all real-valued functions continuous on the interval [a, b] with the function d defined by

$$d(f,g) = \left(\int_a^b f(x) - g(x)^2\, dx\right)^{\frac{1}{2}}$$ is a metric space.

ii) Show that in any metric space (x, d) the intersection of a finite number of open sets is open. 4+2

f) Solve the equation

$$\frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} - 4(x-1)y = 0$$ in series about the point x=1.

g) Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point (0,0) but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

4+2

h) Show that $\int_2^\infty \frac{\cos x}{\log x} dx$ is conditionally convergent.

4. Answer any **three** questions: 15×3=45

a) i) Prove that every closed subset of a compact metric space is compact.

ii) Let l_∞ be the set of all bounded numerical sequences $\{x_n\}$ in which the metric d is defined by

$$d(x, y) = \sup_n |x_n - y_n|, \forall x = \{x_n\}, y = \{y_n\} \in l_\infty.$$

Then show that (l_∞, d) is a complete metric space.

iii) Prove that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy = \frac{1}{2},\ \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx = -\frac{1}{2}.$$

Does the double integral $\iint_R \frac{x-y}{(x+y)^3} dxdy$ exist over R=[0, 1; 0, 1]

4+6+4+1

b) i) Solve :

$$\left(x^2+y^2+yz\right)p+\left(x^2+y^2-xz\right)q=z\left(x+y\right)$$

where $p\equiv\frac{\partial z}{\partial x}$ and $q\equiv\frac{\partial z}{\partial y}$.

ii) Find a complete and singular integrals of $2xz-px^2-2qxy+pq=0$.

iii) Use Laplace transforms to solve the following problem:

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=e^{-3t},\ y(0)=y'(0)=0.$$

5+5+5

c) i) Prove that a bounded function f, having a finite number of points of discontinuity on [a, b] is integrable on [a, b]

ii) A function f is defined on [0, 1] as follows:

$$f(x)=\begin{cases} 0, \text{when } x \text{ is irrational or zero, and} \\ 1/v, \text{when } x \text{ is any non-zero rational number } p/v \text{ with} \\ \quad \text{least positive integers } p \text{ and } v. \end{cases}$$

show that f is integrable on [0, 1] and the value of the integral is zero.

iii) If f is monotone and f, f' and g are all continuous in [a, b] then prove that there exists $\{\in[a,b]$ such that

$$\int_a^b f(x)g(x)dx=f(a)\int_a^{\xi} g(x)dx+f(b)\int_{\xi}^b g(x)dx.$$

5+5+5

d) i) Show that continuous image of a compact set is compact.

ii) Let (X, d) be a metric space. Then show that any disjoint pair of closed sets in X can be separated by disjoint open sets in X.

iii) Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right)dxdy$, where E is the region bounded by the co-ordinate axes and x+y=1 in the first quadrant.

5+5+5

e) i) Solve:

$$z(x+y)\frac{\partial z}{\partial x}+z(x-y)\frac{\partial z}{\partial y}=x^2+y^2.$$

ii) Use Laplace transforms to solve the following problem:

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{-t},\ y(0)=y'(0)=0.$$

iii) Solve the partial differential equation $px + qy = pq$ by Charpits method.

5+5+5

f) i) Prove that a bounded function f is integrable on [a, b] iff for every $\varepsilon > 0$ there exists a partition P of [a, b], such that $U(P, f) - L(P, f) < \varepsilon$.

ii) If f and g are integrable on [a,b] and g keeps the same sign over [a, b], then show that there exists a number μ lying between the bounds of f such that

$$\int_a^b fg\,dx = \mu \int_a^b g\,dx .$$

iii) Prove that lim I_n, where

$I_n = \int_0^\delta \frac{\sin nx}{x} dx, n \in N$ exists and equal to $\pi/2$.

6+3+6

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