U.G. 4th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-8

(Riemann Integration and Series of Functions)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Find the radius of convergence of the following series:

$$x+\frac{(\sqrt{2}x)^2}{2!}+\frac{(\sqrt{3}x)^3}{3!}+\frac{(\sqrt{4}x)^4}{4!}+\ldots$$

b) If the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence R, $0<R<\infty$, then show that the series doesn't converge for any x for which $|x-a|>R$.

c) If a periodic function f of period $2\omega(\omega>0)$ is Riemann integrable on $[-\omega, \omega]$ then show that

$$\int_{-\omega+a}^{\omega+a} f(x)dx=\int_{-\omega}^{\omega} f(x)dx.$$

d) Let a and b $(a<b)$ be two real numbers and n be a positive integer, then prove that $\int_a^b \sin\left(\frac{2n\pi x}{b-a}\right)dx=0$.

e) If f is bounded and integrable on $[-\pi, \pi]$ and if a_n, b_n are its Fourier coefficients, then prove that $\sum_{n=1}^{\infty}(a_n^2+b_n^2)$ converges.

f) Find the series of sines and cosines of multiples of x for the function | cos x | in the interval $[-\pi, \pi]$.

g) Show that the integral $\int_0^{\infty}\frac{\sin x}{x}dx$ is not absolutely convergent.

h) Let a bounded function $f:[a, b]\to\Re$ be Riemann Integrable. Show that for any $\varepsilon>0$

there exists a $\delta > 0$ such that for any partition P of $[a, b]$ with $\|P\| < \delta$ we have $\left|S(P, f) - \int_a^b f(x)\,dx\right| < \delta$, where $S(P, f)$ denotes Riemann sum of f associated with P.

i) Determine the radius of convergence of the power series: $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}$.

j) Suppose that the bounded function $f : [a, b] \to \Re$ has the property that for each rational number x in $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f(x)\,dx \le 0 \le \overline{\int_a^b} f(x)\,dx.$$

k) Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly and absolutely on $\Re$.

l) If g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for a finite set of points in $[a, b]$, then show that f is Riemann integrable and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

m) Test the convergence of the integral $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}\,dx$.

n) Let $g_n : [0, 1] \to \Re$ be defined by

$$g_n(x) = \begin{cases} 0, & x = 0 \\ n, & 0 < x \le \frac{1}{n} \\ 0, & \frac{1}{n} < x \le 1 \end{cases}.$$

If g is the pointwise limit of $\{g_n\}$ on $[a, b]$, show that $\int_0^1 g_n(x)\,dx$ doesn't converge to $\int_0^1 g(x)\,dx$.

o) Show that $\Gamma(n+1) = n\Gamma(n),\ n > 0$.

2. Answer any **four** questions: 5×4=20

a) Find the radius of convergence of the power series $\frac{x^2}{1^2} + \frac{x^2}{2^2} + \frac{x^2}{3^2} + \ldots + \frac{x^2}{n^2} + \ldots$

b) Let $\{f_n\}$ be sequence of differentiable functions defined on $[a, b]$ such that $\{f_n\}$ converges pointwise to the function f defined on $[a, b]$. State a set of sufficient conditions such that f will be differentiable on $[a, b]$ and $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all x in $[a, b]$. Examine the hypothesis and conclusion of the above for the function $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, $x \in R$.

c) Let $f:[a, b] \to \Re$ be differentiable on $[a, b]$ and f' be Riemann integrable on $[a, b]$. Then, prove that $\int_a^b f'(x)\, dx = f(b) - f(a)$.

d) Show that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$, $-\pi \le x \le \pi$. Using this equality show that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

e) Suppose a function $f(x)$ be Darboux integrable on $[a, b]$. Show that there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that $\lim_{n\to\infty}\left[U(f, P_n) - L(f, P_n)\right] = 0$.

f) Show that the following series converges uniformly on $\Re$:

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2+n^4x^2}.$$

Examine whether the derivative of its sum function can be found by term by term differentiation.

3. Answer any **two** questions: 10×2=20

a) i) Show that the following series converges uniformly on $[0, k]$ where $k > 0$, but not on $[0, \infty)$:

$$\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$$

ii) Examine the convergence of the integral

$$\int_0^{\infty} \frac{x \tan^{-1} x}{\left(1+x^4\right)^{\frac{1}{3}}}\, dx.$$

b) i) Let $f:[a, b] \to \Re$ be bounded. Define the upper Darboux sum, lower Darboux sum and Riemann sum of the function $f(x)$. Show that if the function $f(x)$ is Darboux integrable on $[a, b]$ then it is Riemann integrable on $[a, b]$.

ii) Find the Fourier series of $f(x)$ in $[-\pi, \pi]$ where

$$f(x)=\begin{cases} 0, & -\pi \le x < 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$$

c) i) If $f:[a, b] \to \Re$ is continuous and $\int_{\alpha}^{\beta} f(x)\,dx = 0$ for all α, β where $a \le \alpha < \beta \le b$, then show that $f(x) = 0$ on $[a, b]$.

ii) Show that the power series given below is uniformly convergent on [–1, 1]:

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots$$
